

# Communication Learning in Social Networks: Finite Population and the Rates\*

Jianqing Fan<sup>†</sup>

Princeton University

Xin Tong<sup>‡</sup>

MIT

Yao Zeng<sup>§</sup>

Harvard University

## Abstract

Following the Bayesian communication learning paradigm, we propose a *finite population learning* concept to capture the level of information aggregation in any given network, where agents are allowed to communicate with neighbors repeatedly before making a single decision. This concept helps determine the occurrence of effective information aggregation in a finite network and reveals explicit interplays among parameters. It also enables meaningful comparative statics regarding the effectiveness of information aggregation in networks. Moreover, it offers a solid foundation to address, with a new perfect learning concept, long run dynamics of learning behavior and the associated learning rates as population diverges. Our conditions for the occurrence of finite population learning and perfect learning in communication networks are very tractable and transparent.

KEYWORDS: Social networks, Bayesian update, communication, finite population learning, perfect learning, learning rates.

---

\*We thank Daron Acemoglu, Sébastien Bubeck, John Campbell, Emmanuel Farhi, Drew Fudenberg, Matthew Jackson, Gareth James, Philip Reny, Philippe Rigollet, Andrei Shleifer, Alp Simsek, and Yiqing Xing for valuable comments and suggestions. All errors are ours.

<sup>†</sup>Department of Operations Research and Financial Engineering, Princeton University.

<sup>‡</sup>Department of Mathematics, Massachusetts Institute of Technology.

<sup>§</sup>Department of Economics, Harvard University.

# 1 Introduction

The effectiveness of information aggregation has been long and widely recognized as a central theme for good decision making at both individual and aggregate levels. Boosted by the Internet and particularly online social networks, this theme is especially important in communication and decision making in the modern world. People communicate with their friends, through extremely efficient, open and multi-dimensional approaches, in social networks before making specific decisions. In particular, these circumstances of information exchange often involve strategic interactions among people, which call for new modeling techniques beyond mainstream statistics and economics literature, such as the convolution of game theory and graphical models.

Recently, [Acemoglu et al. \(2012a\)](#) provide a fascinating model to study communication in social networks and the implications for information aggregation. They employ a game-theoretic framework to model people’s information aggregation in social networks. They define an intuitive concept of asymptotic learning, which means as the population of a network diverges, the probability that a large fraction of people take “correct” actions converges to one or eventually exceeds a high threshold. Given agents communicate either truthfully or strategically, they establish equilibrium conditions under which asymptotic learning occurs. They also discuss the welfare implications of asymptotic learning, and investigate the impacts of specific types of cost structures and social cliques.

Motivated by the asymptotic learning concept, we ask the following questions. Can we define a good communication learning concept regarding a finite population network? If so, does such learning occur in a given finite social network? What are necessary and sufficient conditions to guarantee such learning? Can we write down clean and tractable rates at which a society achieves long run asymptotic learning? These questions are relevant and important, because it is common practice for people to assess the effectiveness of information aggregation in given organizations, regions or nations. Such assessment regarding finite population networks naturally offers a solid foundation for people to

understand the quality of social learning when the society evolves. As of now, current researchers in social networks have not provided desirable answers to these questions, and previous works called for fresh inputs (Goyal (2009); Acemoglu and Ozdaglar (2010); Jackson (2010)).

Based on an information exchange game in social networks modified from Acemoglu et al. (2012a), we propose a *finite population learning* concept, which captures the level of aggregation of disperse information in any given communication network. In the model, there is an underlying state. People in a social network do not know the underlying state, but they have a common prior on the distribution of the state. After receiving initial private signals related to the underlying state, they exchange information simultaneously in the network, at times specified by a homogenous Poisson process, until taking an irreversible action to exit the network. Upon each person's exit, she makes an estimate of the underlying state. Her payoff depends on the waiting time before making the decision and the expected mean-square error between her estimate and the underlying state. The longer she waits, the more information she gathers and hence the better her estimate is, but the more discounting incurs. Thus, she needs to take a prompt action after obtaining sufficient amount of information in the network.

The newly defined finite population learning concept involves three parameters,  $\epsilon$ ,  $\bar{\epsilon}$ , and  $\delta$  for a given social network  $G_n$  of population size  $n$ ; rigorously, it is called  $(\epsilon, \bar{\epsilon}, \delta)$ -learning. The parameter  $\epsilon$  is the precision under which an agent's decision is considered "correct",  $1 - \bar{\epsilon}$  represents the fraction of agents in the network who make the approximately correct decision, and  $1 - \delta$  represents the probability at which such a fraction of agents make the approximately correct decision. We think of these three parameters as tolerance parameters of finite population learning. To contrast with asymptotically driven concepts,  $(\epsilon, \bar{\epsilon}, \delta)$ -learning is simply referred to as finite population learning in verbal discussions.

We derive necessary and sufficient conditions for the occurrence of finite population learning under any given equilibrium. Intuitively, finite population learning is more likely

to occur when the number of signals an agent obtains under equilibrium is larger, or the tolerances of learning are larger. Interestingly, the impact of the information precisions on finite population learning is ambiguous, which parallels the well-known Hirshleifer effect and subsequent work on the social value of information but stems from a new and different mechanism. We also provide necessary and sufficient conditions for the occurrence of finite population learning under *any equilibrium*, namely, without knowledge of a particular equilibrium.

A straightforward advantage of our conditions is that these conditions lead to meaningful comparative statics regarding the effectiveness of information aggregation in networks. In these conditions, the underlying forces, such as tolerances, information precisions and information-sensitiveness, that shape the effectiveness of information aggregation in a given finite communication network are explicitly displayed in a single formula. Compared to the asymptotic learning results in previous literature, our conditions for finite population learning involve only one equilibrium outcome, which is the number of signals an agent obtains when she exits under equilibrium. More importantly, different from our finite population learning concept in which the total amount of information is fixed, the existing asymptotic learning literature employs an implicit assumption that the total amount of information grows linearly with the population size. Hence, the learning status with respect to a sequence of networks with growing population reflects not only the effectiveness of information aggregation of certain network structures, but also an increased endowment of total information. Our finite population learning concept overcomes this defect and disentangles the effectiveness of information aggregation from the growth of information endowment.

The finite population learning concept enables us to investigate the rate at which a sequence of growing communication networks  $\{G_n\}_{n=1}^\infty$ , which is referred to as a *society*, reaches *perfect learning*. Perfect learning occurs if all communication networks in a society achieve finite population learning under vanishing tolerances as population grows. For example, we say  $\delta$ -perfect learning occurs along society  $\{G_n\}_{n=1}^\infty$  if i).  $(\epsilon, \bar{\epsilon}, \delta_n)$ -learning

occurs for each network  $G_n$  in the society, and ii).  $\delta_n$  goes to zero as  $n$  goes to infinity. The learning rate is characterized by the sequence  $\{\delta_n\}_{n=1}^\infty$ . Clearly, faster learning rate implies perfect learning is reached at a higher quality. It is instructive to distinguish our learning rate concept from the speed of convergence to a pre-defined consensus in existing social learning literature, which mainly concerns about the time towards a consensus in a circumstance where people make repeated decisions and learn from others' previous decisions to help to make their own future decisions. In such a context, the observable sequence of aggregate decisions naturally reveals the dynamics of information aggregation along the time dimension. In our story of direct communication, however, although people communicate with each other repeatedly, they only make a single decision, and different people may go through varying communication rounds before their decisions. This makes the time dynamics of information aggregation largely unobservable, and thus calls for alternative dimensions to look into the information dynamics.

We have given conditions for societies to reach  $\delta$ -perfect learning at a certain desired rate  $\{\delta_n\}_{n=1}^\infty$ . Given a sequence of networks and the associated equilibria, we define an *equilibrium informed agent* as one who obtains an unbounded number of signals as the population size goes to infinity. The  $\delta$ -perfect learning occurs if almost all agents in the society are equilibrium informed. Moreover, without involving any equilibrium, we define a *socially informed agent* (roughly) as one who has an unbounded number of neighbors in a finite distance as the population goes to infinity. The  $\delta$ -perfect learning occurs if almost all agents are socially informed. We also explicitly explore the achievable fastest learning rate for perfect learning in a given society. Under some circumstances, achievable learning rate could be in the exponential order. This implies that a society with growing population might achieve a desirable level of finite population learning very quickly.

RELATION TO LITERATURE. Our work lies in the category of Bayesian social learning in social networks, in which decision makers in a social network update their information

according to the Bayes' rule. General Bayesian social learning is divided into two sub-categories, namely Bayesian observational learning and Bayesian communication learning. In Bayesian observational learning, agents observe past actions of their neighbors. From these observed actions, agents update their beliefs and make inferences. Herd behavior is a very typical consequence of observational learning. In literature, [Banerjee \(1992\)](#), [Bikhchandani et al. \(1992\)](#) and [Smith and Sorensen \(2000\)](#) are early attempts to model herd effects through Bayesian observational learning. [Banerjee and Fudenberg \(2004\)](#) and [Smith and Sorensen \(2008\)](#) relax the assumption of full observation network topology and study Bayesian observational learning with sampling of past actions. Recently, [Acemoglu et al. \(2011\)](#) and [Muller-Frank \(2012\)](#) investigate how detailed network structures could add new interesting insights.

Our work belongs to Bayesian communication learning, which means that agents cannot directly observe actions of others but can communicate with each other before making a decision. Consequently, agents update their beliefs and make inferences based on the information given by others. New interesting considerations arise in Bayesian communication learning; for example, agents may not want to truthfully reveal their information to others through communication. [Crawford and Sobel \(1982\)](#) pioneers the research in strategic communication, and [Acemoglu et al. \(2012a\)](#) is an interesting piece that looks into how communication learning shapes information aggregation in social networks. Other works such as [Galeotti et al. \(2011\)](#) and [Hagenbach and Koessler \(2010\)](#) also study strategic communication in social networks, but their focus is not on information aggregation.

There is a branch of literature that applies various non-Bayesian updating methods to investigate information aggregation and social learning. [DeGroot \(1974\)](#) develops a tractable non-Bayesian learning model which is frequently employed in research of social networks today. Essentially, the DeGroot model is pertaining to observational learning, in which agents make today's decisions by taking the average of neighbors' beliefs revealed in their decisions yesterday. [DeGroot \(2003\)](#) and [Golub and Jackson \(2010, 2012a,b,c\)](#) apply

the DeGroot model to financial networks and general social networks, respectively. By a field experiment, [Mobius et al. \(2010\)](#) compares a non-Bayesian model of communication with a model in which agents communicate their signals and update information based on Bayes' rule. Their evidence is generally in favor of the Bayesian communication learning approach.

Our paper is most related to [Acemoglu et al. \(2012a\)](#). Compared to their work, we employ a simplified framework for network communication and exploit more undeveloped mechanisms. In particular, we mainly focus on the effect of social learning and information aggregation in finite population communication networks. This allows for clear comparative statics with respect to learning, and for discussion on the rates of learning as the population increases. As of now, researchers have not provided desirable results in finite population communication network as well as results regarding learning rates. To the best of our knowledge, our work is the first attempt to address these questions with clear answers.

Our work is also related to [Golub and Jackson \(2012a,b,c\)](#), in particular on the investigation of learning rate. [Golub and Jackson \(2012a,b,c\)](#) employ the DeGroot model to analyze the impacts of homophily in social networks, which refers to the tendency of agents to associate relatively more with those who are similar to them, on the learning rate in the context of observational learning. Our results of learning rate are different from theirs in two aspects. First, our focus is on Bayesian communication learning rather than non-Bayesian observational learning. Second, as discussed before, our concept of learning rate is based on perfect learning as the population in networks diverges, rather than the time towards a consensus in their model. An appealing feature of [Golub and Jackson \(2012a,b,c\)](#) is that their results of learning rate are based on certain statistics of networks rather than the full network structures, which could lead to potentially more empirical traction.

We would also like to relate this work to social network papers in existing statistics literature. The larger part of those papers are based on graphical models, which are ideal

to describe structural formation. Rather than providing a list of state-of-the-art contributions, we refer interested readers to [Newman \(2010\)](#) and [Kolaczyk \(2009\)](#), which might serve as a broad introduction to the field. Our work supplements structural modeling with human behavior modeling through game theory. Such model enrichment is necessary for some specific objectives; for example, we will see that strategic interaction and contextual information that sit outside graphical models are crucial to determine the final information aggregation status. Also, our theoretical results are in the same spirit of the finite sample results in the statistical learning theory, such as the Vapnik-Chervonenkis inequality. Such results with a clear characterization of strategic interactions may have potential to expand the scope of the finite sample approach beyond statistical learning theory.

The rest of the paper is organized as follows. Section 2 introduces the information exchange game and characterizes its equilibrium. New finite population learning concept is proposed in Section 3. Section 4 discusses dynamics of learning and addresses learning rates explicitly. In the final section, we discuss possible directions for further research. All proofs are in the supplementary materials.

## 2 The Model

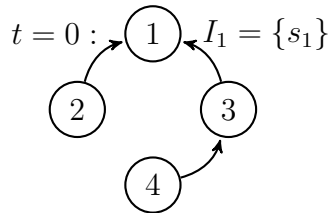
In this section, we present our model of information exchange in social networks, which is closely related to [Acemoglu et al. \(2012a\)](#), but has different focus. In this model, people, formally called as agents, are organized in some network structure. Each agent has her initial information. Agents are able to solicit information from their neighbors through communication, restricted by the network structure and a communication clock. The communication clock defines the times at which each agent is able to communicate with others. At each round of communication, agents are obliged to transmit truthfully *all* information they have to their neighbors in the network. By such communication, the information set of an agent can become larger as time evolves. With the help of her initial



and acquired information, every agent is able to make a decision and exit. An exit strategy is needed due to the time value of information content. After exit, an agent would not have any incentive to further acquire information from neighbors, but she is still obliged to transmit all her information to others in the next round of communication. Through certain measure of agents' decisions, we are further able to characterize the quality of learning and information aggregation.

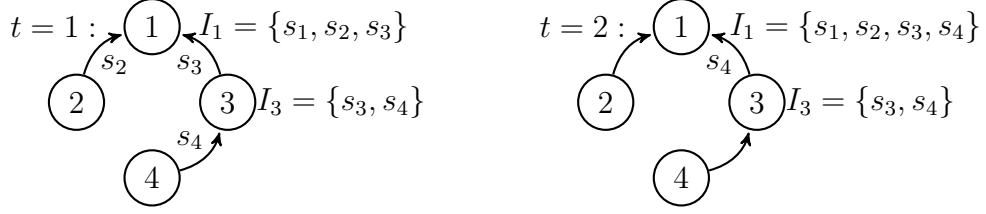
We make the following assumptions to simplify the analysis and focus on a concept of finite population learning, which will be rigorously defined in the next section. First, we assume mandatory communication, which means that no agent holds her information to herself. When communication times arrive, an agent has to send all her information set to all of her direct neighbors. Second, we assume truthful communication, which means whenever an agent sends information, she has to send unmanipulated information, whether it is her own private information or obtained information originated from other agents. We will first analyze communication and information aggregation in a given finite population network, and then consider the limit as the population grows to infinity. In this course, we assume that existing links are kept when a network grows.

Before formal definition of the game, we would like to illustrate how information flows with an example. For simplicity, suppose there are four agents in the network below. At time  $t = 0$ , each agent  $i$  has some private signal  $s_i$ , which captures her initial information. So the total information endowment in the system is  $\{s_1, s_2, s_3, s_4\}$ . Communication occurs at  $t = 1, 2$ . Due to the structure of the graph in our example, there is no need to consider beyond the second communication round, since no additional information will be communicated further.



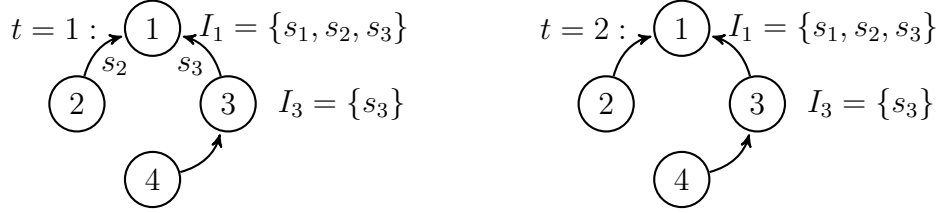
We will study two cases, and focus on agent 1's information set  $I_1$ . In the first case,

suppose no agent exits after time  $t = 0$ . So the information flow is as follows:



After the first round of communication, i.e.,  $t = 1$ , agent 1 has signals  $\{s_1, s_2, s_3\}$ . Also note that at this time agent 3 has  $\{s_3, s_4\}$ . At  $t = 2$ , agent 3 sends the newly grabbed signal  $s_4$  to agent 1. So agent 1's information set enriches to  $\{s_1, s_2, s_3, s_4\}$ .

In the second case, suppose agent 3 exits after time  $t = 0$ , then she is still obliged to send all her signals (in this case, only her private signal) she acquires to neighbors, but she does not have any incentive to receive others' signal. Therefore, the information flow is as follows.



Note that as agent 3 does not receive signal from agent 4 at  $t = 1$ , she does not have any new information to send to agent 1 at the second communication round. Therefore, agent 1's information set is still  $\{s_1, s_2, s_3\}$  at  $t = 2$ . By contrasting the two cases in this toy example, we see that agents' decisions affect the information flow in the network.

Now we formally introduce the information exchange game. Suppose we are interested in a social network with agents  $\mathcal{N}^n = \{1, 2, \dots, n\}$ . To model communication in the network, we organize these agents in a directed graph  $G_n = (\mathcal{N}^n, \mathcal{E}^n)$ , in which each node  $i \in \mathcal{N}^n$  represents an agent. We allow directed graphs to have multi-edges, so that two agents can communicate to each other. An ordered pair  $(j, i) \in \mathcal{E}^n$  means agent  $j$  can send information to agent  $i$  directly. The goal of every agent is to estimate  $\theta \in \mathbb{R}$ , which represents an underlying state of the world. Agents' knowledge of  $\theta$  is captured by a

normally distributed common prior  $\theta \sim N(0, 1/\rho)$ . At time  $t = 0$ , agent  $i$  is endowed with her private signal  $s_i = \theta + z_i$ . All  $z_i \sim N(0, 1/\bar{\rho})$  are independent and they are also independent of  $\theta$ . The distributions of  $z_i$ 's are common knowledge and so is the network architecture. Our results are not affected if the means of  $\theta$  and  $z_i$  are changed to non-zero values.

In this network, agents exchange their information as follows. Suppose agents live in a world with continuous time  $t \in [0, \infty)$ . Waiting induces a common exponential discount of the payoff with rate  $r > 0$ . Instead of communicating at fixed times, all agents communicate simultaneously at some points in time that follow a homogeneous Poisson process with rate  $\lambda > 0$ , which is independent of  $\theta$  and  $z_i$ . This Poisson clock is also common knowledge. After communication, agents update beliefs according to the Bayes' rule. For example, the posterior distribution of  $\theta$  on  $k$  distinct signals is Gaussian with precision  $\rho + k\bar{\rho}$ . So more private information, i.e., a higher  $k$ , will increase the precision and lead to a better estimate. Hence, there is a natural trade-off between waiting to get more information and acting earlier to reduce the discount of information value, which makes an optimal stopping problem for each agent  $i$ . We call the incentive to get more information *information effect*, and the incentive to act earlier *discount effect*. In this course, at any given time  $t$ , each agent  $i$  either makes an estimate  $x_i$  of the fundamental state of the world  $\theta$ , or "wait" for more information. Just as illustrated in the four agents' example, we assume that after agents make estimate and exit, they do not receive new information, but they continue to transmit information that they have already obtained when new rounds of communication take place.

We introduce a few more notations to facilitate the discussion. Let  $I_{i,t}^n$  denote the information set of agent  $i$  at time  $t$ . We next specify the payoff structure and the optimization problem faced by agents. Suppose agent  $i$  takes action  $x_i$  at time  $t$  when the realization of the underlying state is  $\theta$ , then her instantaneous payoff of taking an action  $x_i$  is

$$u_i^n(x_i) = \psi - (x_i - \theta)^2,$$

where  $\psi$  is a real-valued constant that captures the information sensitiveness of the decision problem, which we will elaborate later. At time  $t$  with information set  $I_{i,t}^n$ , agent  $i$ 's optimal expected instantaneous payoff of taking an action before discounting is

$$U_{i,t}^n(I_{i,t}^n) = \max_{x_i} \mathbb{E}(u_i^n(x_i) | I_{i,t}^n).$$

It is easy to see that agent  $i$ 's optimal estimate is  $x_{i,t}^{n,*} = \mathbb{E}[\theta | I_{i,t}^n]$  if she decides to act at time  $t$ . Thanks to the normality assumption of the fundamental  $\theta$  and signals  $\{s_i\}_{i=1}^n$ , the optimal expected instantaneous payoff of agent  $i$  taking an action after observing  $k$  distinct signals can be calculated explicitly:

$$\mathbb{E}[\psi - (x_{i,t}^{n,*} - \theta)^2 | I_{i,t}^n] = \psi - \frac{1}{\rho + \bar{\rho}k}. \quad (2.1)$$

At any time  $t$  with information set  $I_{i,t}^n$ , before trying to make a best estimate and exit, agent  $i$  has to make a decision about whether to exit. To facilitate the analysis, we first assume that any agent can obtain non-negative payoff upon her exit. This assumption will be formally characterized after we define the equilibrium. As a result, due to discount in time, each agent should make an estimate and exit precisely at a finite time, and especially, at a time instantaneously after communications take place. Moreover, each agent would only get finite number of signals even if they waited forever, because there are in total  $n$  signals  $\{s_i\}_{i=1}^n$  in the network. Therefore, we actually only need to consider strategy profiles in which every agent exits at a finite communication round, rather than at any arbitrary time. Denote by  $l^n = (l_1^n, \dots, l_n^n)$ , where each  $l_i^n$  is agent  $i$ 's communication round before exit. Throughout the paper, we use  $l_{-i}^n$  to denote  $l^n$  without the component  $l_i^n$ . Let  $\tau_k$  be the physical time until  $k$  rounds of communication. Agent  $i$ 's payoff for choosing action  $l_i^n$  is

$$U_i^n(l_i^n, l_{-i}^n) = \mathbb{E} \left\{ e^{-r\tau_{l_i^n}} \max_{x_i} \mathbb{E}[\psi - (x_i - \theta)^2 | I_i^n(l^n)] \right\},$$

where  $I_i^n(l^n)$  is agent  $i$ 's information set upon exit, which depends on other agents' exit strategies  $l_{-i}^n$ . By (2.1) and the exponential waiting time of the Poisson clock, we have

$$U_i^n(l_i^n, l_{-i}^n) = \bar{r}^{l_i^n} \left( \psi - \frac{1}{\rho + \bar{\rho} k_i^{n, l^n}} \right),$$

where  $\bar{r} = \lambda/(\lambda + r)$  and  $k_i^{n, l^n}$  is the number of signals agent  $i$  get upon exit if every agent acts according to  $l^n$  in the network  $G_n$ . With this reduction, the following complete information static game will be considered.

**Definition 1** *The information exchange game  $\Gamma_{\text{info}}(G_n)$  is a triple  $\{\mathcal{N}^n, \mathcal{L}^n, \mathcal{U}^n\}$ , in which*

- (a)  $\mathcal{N}^n$  is the set of agents, i.e.,  $\mathcal{N}^n = \{1, 2, \dots, n\}$ ;
- (b)  $\mathcal{L}^n$  is the collection of agents' strategy spaces. For any agent  $i \in \mathcal{N}^n$ , her strategy space  $L_i^n \in \mathcal{L}^n$  is a finite set

$$L_i^n = \{0, 1, 2, \dots, (L_i^n)_{\max}\},$$

where  $(L_i^n)_{\max} = \max_{j \in G_n} \{\text{length of shortest path from } j \text{ to } i\}$ ;

- (c)  $U_i^n \in \mathcal{U}^n$  is the payoff function for agent  $i$ :

$$U_i^n(l_i^n, l_{-i}^n) = \bar{r}^{l_i^n} \left( \psi - \frac{1}{\rho + \bar{\rho} k_i^{n, l^n}} \right). \quad (2.2)$$

We consider pure-strategy Nash equilibria of this game. As an agent's payoff gain from waiting is weakly larger (i.e., no smaller than) when other agents also wait more rounds, the information exchange game is a supermodular game. The following result is a direct application of [Topkis \(1979\)](#), which guarantees the existence of a pure-strategy Nash equilibrium in supermodular games.

**LEMMA 1** *The information exchange game  $\Gamma_{\text{info}}(G_n)$  has at least one pure-strategy Nash equilibrium.*

We denote a pure-strategy Nash equilibrium of the game by  $\sigma^{n,*}$ , and the set of all pure-strategy Nash equilibria by  $\Sigma^{n,*}$ . We further denote by  $l_i^{n,\sigma^*}$  the communication steps after which agent  $i$  exits under equilibrium  $\sigma^{n,*}$ , and denote by  $k_i^{n,\sigma^*}$  the number of distinct signals agent  $i$  has obtained when she exits under equilibrium  $\sigma^{n,*}$ . In order to make sure that every agent  $i$  gets non-negative payoffs and exits ultimately in the initial strategic circumstance of information exchange, we focus on information exchange games and associated equilibria that satisfy the following assumption in the rest of this section.

**Assumption 1**  $\psi[\rho + \bar{\rho}(k_i^{n,\sigma^*})_{\max}] \geq 1$  for all agent  $i$ , where  $(k_i^{n,\sigma^*})_{\max}$  is the maximum number of signals agent  $i$  can get if all other agents choose their exit steps according to  $\sigma^{n,*}$ .

The parameter  $\psi$  captures the information sensitiveness of the decision problem. Interestingly, the information sensitiveness of the decision problem is not monotone in  $\psi$ . When  $\psi$  takes negative or very small positive value, agents would like to wait forever to discount payoff to zero, in which case the decision problem is information irrelevant. When  $\psi$  is large enough, information is relevant. Specifically, when  $\psi$  is moderate, information effect dominates, and thus the decision problem is more information sensitive; while when  $\psi$  is large, the discount effect dominates, and thus the decision problem is less information sensitive.

Now we provide an example of the network game and its equilibrium. On the four-agent graph displayed previously, suppose  $\lambda = r$ ,  $\psi = 1$  and  $\rho = \bar{\rho} = \frac{1}{2}$ . The decision problem for agents 2 and 4 are simple. They should exit right away because they will not get any new signals due to graph structure, but incur discounting penalty should they not act promptly. The payoff matrix for agent 1 (row) and 3 (column) is as follows, in which the first and the second value in each cell are respectively the payoffs of agent 1 and agent 3 [see (2.2)].

		Agent 3	
		0 Step	1 Step
Agent 1	0 Step	0, 0	0, $\frac{1}{6}$
	1 Step	$\frac{1}{4}$ , 0	$\frac{1}{4}$ , $\frac{1}{6}$
	2 Step	$\frac{1}{8}$ , 0	$\frac{3}{20}$ , $\frac{1}{6}$

There is one equilibrium of the game. In this equilibrium, agents 2 and 4 exits immediately after they receive their private signals, while agent 1 and agent 3 exit after the first communication round.

Before proceeding to discuss information aggregation or learning status, we briefly discuss the equilibrium outcomes of the strategy game in Definition 1. This reduced game is a complete information static game, which involves no uncertainty. However, the uncertainties in the fundamental and in the communication clock were abstracted out through taking expectations, which results in the deterministic payoff function (2.2). Therefore the two equilibrium outcomes,  $l_i^{n,\sigma^*}$  and  $k_i^{n,\sigma^*}$ , both deterministic, characterize the strategic interactions of information exchange among agents in the initial circumstance. This enables us to characterize a learning status by focusing only on such equilibrium outcomes.

We can perform the following comparative statics of the number of signals agent  $i$  obtains under equilibrium  $k_i^{n,\sigma^*}$ . Intuitively,  $k_i^{n,\sigma^*}$  is larger when the discount rate is smaller or the Poisson clock is faster. It is also larger when the precision of public information  $\rho$  is lower or the decision problem is more information sensitive. However, the precision of private information  $\bar{\rho}$  has ambiguous impact on  $k_i^{n,\sigma^*}$ , because an increase in the precision of private information has two conflicting effects. It increases not only the relative quality of the private signal at hand, which prompts an agent to exit earlier, but also the relative information content of her neighbors' private signals, which in turn encourages her to wait. The former effect is stronger when the precision of public information is higher, while the latter is stronger when the precision of public information is lower. The discussions in this paragraph can be formalized against mathematical rigor, but the game-theoretic technicality involved is beyond the scope of this paper.

Finally, we also remark that the role of  $k_i^{n,\sigma^*}$  in our paper is similar to the influence vector  $v$  in [Acemoglu et al. \(2012b\)](#). The quantity  $k_i^{n,\sigma^*}$  will play a central role in the next sections.

### 3 Finite Population Learning

In this section, we measure the level of information aggregation in any given communication network. Related recent research on learning in social networks focuses on asymptotic learning, which means that as the fraction of agents taking the correct action converging to one as the population of the social network grows large ([Acemoglu et al. \(2011, 2012a\)](#)). However, as discussed in [Acemoglu and Ozdaglar \(2010\)](#), people are also interested in the information dynamics away from long run limit. In pursuing this goal, a new concept of learning in social networks is introduced.

**Definition 2** *Given a social network  $G_n$ , the information exchange game  $\Gamma_{\text{info}}(G_n)$  and an equilibrium profile  $\sigma^{n,*}$ , for a triple  $(\varepsilon, \bar{\varepsilon}, \delta)$ , we say  $G_n$  achieves  $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning under  $\sigma^{n,*}$  if*

$$\mathbb{P}_{\sigma^{n,*}} \left( \frac{1}{n} \sum_{i=1}^n (1 - M_i^{n,\varepsilon}) \geq \bar{\varepsilon} \right) \leq \delta,$$

where  $M_i^{n,\varepsilon} = \mathbf{1}(|x_i - \theta| \leq \varepsilon)$ ,  $x_i$  is agent  $i$ 's optimal action upon exit, and  $\mathbb{P}_{\sigma^{n,*}}$  denotes the conditional probability given  $\sigma^{n,*}$ .

In this definition, the parameter  $\varepsilon$  sets the precision on what the approximately correct decision is for individual agents,  $1 - \bar{\varepsilon}$  controls the fraction of agents who make the approximately correct decision, and  $1 - \delta$  represents the probability at which such a high fraction of agents make the approximately correct decision. In particular, we highlight the difference between  $\varepsilon$  and  $\bar{\varepsilon}$ , because these two parameters capture different tolerances. Concretely,  $\varepsilon$  is at the individual level while  $\bar{\varepsilon}$  is at the aggregate level.

A natural question to ask is whether such finite population learning occurs in a given communication network. If so, under what conditions? The following proposition provides



a necessary condition and a sufficient condition for  $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning in a given social network under any equilibrium profile. When there is no confusion, we refer to the information exchange game  $\Gamma_{\text{info}}(G_n)$  simply as  $G_n$ . Denote by  $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$  the error function of the standard normal distribution.

**Proposition 1** *For a given social network  $G_n$  under any equilibrium  $\sigma^*(= \sigma^{n,*})$ ,*

(a)  *$(\varepsilon, \bar{\varepsilon}, \delta)$ -learning does not occur if*

$$\frac{1}{n} \sum_{i=1}^n \text{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right) < (1 - \bar{\varepsilon})(1 - \delta). \quad (3.1)$$

(b)  *$(\varepsilon, \bar{\varepsilon}, \delta)$ -learning occurs if*

$$\frac{1}{n} \sum_{i=1}^n \text{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right) \geq 1 - \bar{\varepsilon} \delta. \quad (3.2)$$

This proposition provides clear conditions for the occurrence of finite population learning. Our conditions are more operative and transparent than their asymptotic counterparts in previous literature. Specifically, our conditions only require one equilibrium outcome  $k_i^{n,\sigma^*}$ , and the set  $\{k_i^{n,\sigma^*}\}_{i=1}^n$  is directly induced by an equilibrium  $\sigma^{n,*}$  in a communication network  $G_n$ . Hence, conditions (3.1) and (3.2) not only allow us to investigate the effect of learning in a given communication network, but also offer a more interpretable link between the communication equilibrium and its corresponding information aggregation status.

Conditions (3.1) and (3.2) also allow us to untangle the interplay among parameters. For example, we are able to answer the following question. Given the tolerances  $\varepsilon, \bar{\varepsilon}, \delta$  and the information precisions  $\rho$  and  $\bar{\rho}$ , how does the change of  $k_i^{n,\sigma^*}$  affect the occurrence of finite population learning in a given social network  $G_n$ ? When  $k_i^{n,\sigma^*}$ 's are sufficiently small to validate condition (3.1), finite population learning does not occur. Similarly, when most of  $k_i^{n,\sigma^*}$ 's are sufficiently large so that the condition (3.2) is satisfied, finite population

learning occurs. Similar marginal interpretations also apply to parameters  $\varepsilon$ ,  $\bar{\varepsilon}$ ,  $\delta$ ,  $\rho$  and  $\bar{\rho}$ . Generally, finite population learning in a given social network  $G_n$  is more likely to occur when the equilibrium induces larger numbers of signals obtained by agents. It is also more likely to occur when the tolerances and the information precisions are higher. As interplays among the parameters  $\varepsilon$ ,  $\bar{\varepsilon}$ ,  $\delta$ ,  $\rho$ ,  $\bar{\rho}$  and  $k_i^{n,\sigma^*}$  are clear through (3.1) and (3.2), the two conditions provide various comparative statics that help us better understand learning in different social circumstances. Since the total amount of information is fixed in any finite population network, these comparative statics indeed disentangle the effectiveness of information aggregation from the endowment of information, so that the net effect of information aggregation is transparent.

It is interesting to note that  $(1 - \bar{\varepsilon})(1 - \delta) < 1 - \bar{\varepsilon}\delta$  for any  $0 < \bar{\varepsilon}, \delta < 1$ . This gap indicates that failure of condition (3.1) does not necessarily lead to condition (3.2), and vice versa. Two perspectives help understand this gap. First, we use Markov's inequality to get tractable forms of the necessary and the sufficient conditions. Sharper inequalities may lead to weaker conditions and thus probably fill a part of the gap, but they are likely to make these conditions intractable and less transparent. Secondly and more importantly, as we discussed above, conditions (3.1) and (3.2) involve equilibrium outcomes in a clean and simple formula. The cost for enjoying this clarity is that we did not fully utilize  $\{k_i^{n,\sigma^*}\}_{i=1}^n$ .

Also, a beauty of symmetry arises in our necessary and sufficient conditions for finite population learning. The parameters  $\bar{\varepsilon}$  and  $\delta$  are completely interchangeable in these conditions, which was not expected as they capture tolerances in different categories. On the other hand, in our two conditions, parameter  $\varepsilon$  stands in a position that is unchangeable with  $\bar{\varepsilon}$  and  $\delta$ , which hints that  $\varepsilon$  and  $\bar{\varepsilon}$  play different roles in finite population learning.

Conditions (3.1) and (3.2) have powerful implications. The next corollary establishes a necessary condition and a sufficient condition without equilibrium outcomes. The proof is straightforward, but the results are non-trivial.

**COROLLARY 1** *For any social network  $G_n$  and any equilibrium  $\sigma^{n,*}$ ,*

*(a)  $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning does not occur if*

$$\operatorname{erf}\left(\varepsilon\sqrt{\frac{\rho + \bar{\rho}n}{2}}\right) < (1 - \bar{\varepsilon})(1 - \delta). \quad (3.3)$$

*(b)  $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning occurs if*

$$\operatorname{erf}\left(\varepsilon\sqrt{\frac{\rho + \bar{\rho}}{2}}\right) \geq 1 - \bar{\varepsilon}\delta. \quad (3.4)$$

Corollary 1 follows from the fact that  $1 \leq k_i^{n,\sigma^*} \leq n$ . It is interesting because under some circumstances, we can determine the occurrence of finite population learning without knowing either the structure of the social network or the equilibrium. Intuitively, if any one parameter of the tolerances, information precisions or population size is too low, such that the condition (3.3) is satisfied, we may conclude that finite population learning does not occur no matter how effective the communication network is organized. Conversely, if any one of the tolerances or information precisions is sufficiently large such that condition (3.4) holds, we know that finite population learning surely occurs even if all agents are isolated.

Finally, as the information exchange game exhibits strategic complementarity, it is expected that multiple equilibria might emerge under some circumstances. An interesting perspective in investigating finite population learning is to measure the effect of learning against multiple equilibria. We provide the following generalized (conservative) version of finite population learning to accommodate multiple equilibria without equilibrium selection.

**Definition 3** *Denote by  $\Sigma^{n,*} = \{\sigma^{n,*}\}$  the set of equilibria of  $\Gamma_{\text{info}}(G_n)$ . The  $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning occurs if*

$$\sup_{\sigma^{n,*} \in \Sigma^{n,*}} \mathbb{P}_{\sigma^{n,*}}\left(\frac{1}{n} \sum_{i=1}^n (1 - M_i^{n,\varepsilon}) \geq \bar{\varepsilon}\right) \leq \delta.$$

This definition offers a conservative standard to evaluate finite population learning in the sense that the least favorable equilibrium determines the learning status. When  $\Sigma^{n,*}$  is a singleton, the above definition reduces to Definition 2. The proof of Proposition 1 can be recycled to derive the next corollary.

**COROLLARY 2** *Given an information exchange game  $\Gamma_{\text{info}}(G_n)$ ,*

(a)  *$(\varepsilon, \bar{\varepsilon}, \delta)$ -learning does not occur if*

$$\min_{\sigma^{n,*} \in \Sigma^{n,*}} \frac{1}{n} \sum_{i=1}^n \text{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right) < (1 - \bar{\varepsilon})(1 - \delta).$$

(b)  *$(\varepsilon, \bar{\varepsilon}, \delta)$ -learning occurs if*

$$\min_{\sigma^{n,*} \in \Sigma^{n,*}} \frac{1}{n} \sum_{i=1}^n \text{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right) \geq 1 - \bar{\varepsilon} \delta.$$

## 4 Perfect Learning and the Rates

Based on the analysis of finite population learning, we consider the effect of information aggregation and learning as population in communication networks grows. Our approach to address the limiting behavior of learning is different from asymptotic learning in existing literature (Acemoglu et al. (2011, 2012a)). In particular, we highlight finite population learning as the foundation of asymptotic learning. Consequently, we are able to check learning status all along the path to the limit, and the probabilistic tolerance parameters naturally induce learning rates. This concept of learning rate is different from what is employed in Golub and Jackson (2012a,b,c) that focuses on the time dimension.

### 4.1 Perfect Learning

Recall that we have three tolerance parameters  $\varepsilon$ ,  $\bar{\varepsilon}$ , and  $\delta$  that define  $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning. To inquire the limiting behavior in a society  $\{G_n\}_{n=1}^\infty$ , where existing links are kept when

networks grow, we can focus on one parameter at a time, keeping the other two fixed. The following definition introduces  $\delta$ -perfect learning on a given society  $\{G_n\}_{n=1}^\infty$ .

**Definition 4** *We say  $\delta$ -perfect learning occurs in society  $\{G_n\}_{n=1}^\infty$  under equilibria  $\{\sigma^{n,*}\}_{n=1}^\infty$  if there exists a vanishing positive sequence  $\{\delta_n\}_{n=1}^\infty$  such that  $(\varepsilon, \bar{\varepsilon}, \delta_n)$ -learning occurs in  $G_n$  under its associated  $\sigma^{n,*}$  for all  $n$ .*

Compared to the perfect asymptotic learning concept in [Acemoglu et al. \(2012a\)](#), our definition of perfect learning is both stronger and more general for the following reasons. First, we require the networks in the society to achieve learning not only in the limit but also all along the path towards the limit. Second, by focusing on different parameters  $\varepsilon$ ,  $\bar{\varepsilon}$  and  $\delta$ , we could address three different kinds of asymptotic learning. As discussed in the previous section, these three parameters exhibit different impacts on finite population learning, so that they can play different roles in perfect learning. Third and most importantly, this definition allows us to investigate learning rates, which is the focus of the next subsection.

In the following, we will derive two sufficient conditions for  $\delta$ -perfect learning. The first condition, stated as [Proposition 2](#), relies on the equilibrium outcome  $k_i^{n,\sigma^*}$ . The second condition, stated as [Proposition 3](#), relies only on the formation of the society. To deliver the first sufficient condition, we define an *equilibrium informed agent* in a society.

**Definition 5 (Equilibrium Informed Agent)** *For agent  $i$  in a given society  $\{G_n\}_{n=1}^\infty$ , she is equilibrium informed with respect to  $\{G_n\}_{n=1}^\infty$  under equilibria  $\{\sigma^{n,*}\}_{n=1}^\infty$  if*

$$\lim_{n \rightarrow \infty} k_i^{n,\sigma^*} = \infty.$$

An agent has equilibrium informed status means that she enjoys increasing information advantage as population grows. The next proposition offers a sufficient condition for  $\delta$ -perfect learning. In a similar spirit, we have a more general sufficient condition, [Lemma 3](#),

in the supplementary materials. The proof of Proposition 2 is omitted as it is a corollary to Lemma 3.

**Proposition 2** *The  $\delta$ -perfect learning occurs in a society  $\{G_n\}_{n=1}^\infty$  under equilibria  $\{\sigma^{n,*}\}_{n=1}^\infty$  if*

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\text{EI}^{n,*}| = 1,$$

*where  $\text{EI}^{n,*}$  the set of equilibrium informed agents in the network  $G_n$  under equilibrium  $\sigma^{n,*}$ .*

Proposition 2 states that perfect learning occurs when almost all agents are equilibrium informed. This is consistent with the idea of social learning that successful learning allows individuals to have sufficient information to make a good decision, and that such individuals represent an overwhelming proportion of the society. We can have such a transparent condition because our perfect learning concept is powered by finite population learning, a sufficient condition of which only involves one set of equilibrium variables:  $\{k_i^{n,\sigma^*}\}$ .

Next we consider the second sufficient condition that relies only on formation of the society. To streamline the presentation in the main texts, we assume that each agent enjoys a positive payoff even if she exits at the beginning. From (2.1), this is equivalent to the following assumption.

**Assumption 2**  $(\rho + \bar{\rho})\psi > 1$ .

We will hold this assumption for the rest of this section. In the supplementary materials, we relax this assumption and discuss all possible cases.

Before looking into the next sufficient condition for perfect learning, we first point out an important observation which states that, although the number of signals an agent gets in equilibrium may diverge to infinity with growth of the communication network, the equilibrium communication steps will not increase unboundedly.

LEMMA 2 Under Assumption 2, for any agent  $i$ , the communication rounds she optimally experiences before taking an action in any social network  $G_n$  is bounded from above by a constant independent of  $n$ . Mathematically,

$$l_i^{n,\sigma^*} \leq l_i^n < \min \left\{ (L_i^n)_{\max}, \ln \left( 1 - \frac{1}{(\rho + \bar{\rho})\psi} \right) / \ln \bar{r} \right\}, \quad (4.1)$$

in which  $l_i^n$  stands for the optimal communication rounds for agent  $i$  given that other agents wait until the maximum allowable step, and  $(L_i^n)_{\max}$  is the maximum length of all paths ended with  $i$  in  $G_n$ .

A more general version (without Assumption 2) of Lemma 2 with its associated proof is included in the supplementary materials as Lemma 4. A key idea behind Lemma 2 is that after agent  $i$  gets sufficiently large number of signals within some finite communication steps, even expecting infinite number of signals does not justify the discount of further waiting. The intuition is that for well connected agents, they will get sufficient information after a few communication rounds to make a decision, whereas for the not well connected agents, waiting too long discounts their information value. From condition (4.1), we see that the upper bound is exclusively determined by parameters of the information exchange game.

Lemma 2 plays an important role in shaping our next sufficient condition that bypasses equilibrium and directly links perfect learning to network formations. Recall Proposition 2 which states that almost all agents'  $k_i^{n,\sigma^*} \rightarrow \infty$  is sufficient for perfect learning. On the other hand, from Lemma 2 we know that no agent has an optimal unbounded communication step  $l_i^{n,\sigma^*}$ . By combining the two observations, we know the only possibility to validate Proposition 2 is that almost all agents get unbounded number of signals within finite communication steps. This consideration leads to our following definition of a *socially informed agent*.

**Definition 6 (Socially Informed Agent)** For each agent  $i$  in a given society  $\{G_n\}_{n=1}^\infty$ , let  $L_i = \min\{l_0 \in \mathbb{N} : \lim_{n \rightarrow \infty} |B_{i,l_0}^n| = \infty\}$ , where  $B_{i,l}^n$  is the set of agents

in  $G_n$  whose shortest path to  $i$  has length at most  $l$ . Agent  $i$  is socially informed with respect to  $\{G_n\}_{n=1}^\infty$  if  $L_i$  is finite, and if there exists  $N \in \mathbb{N}$  such that for  $n \geq N$ , we have

$$\psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_i}^n|} > 0, \quad (4.2)$$

and

$$\bar{r}^{L_i} \left( \psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_i}^n|} \right) > \bar{r}^l \left( \psi - \frac{1}{\rho + \bar{\rho}|B_{i,l}^n|} \right) \text{ for all } 0 \leq l < L_i. \quad (4.3)$$

Moreover, we denote by  $SI^n$  the set of socially informed agents in the network  $G_n$ .

In Definition 6, condition (4.2) is automatically satisfied in view of Assumption 2. Intuitively, a socially informed agent can be reached by a large number of neighbors after some finite communication steps  $L_i$ . Furthermore, condition (4.3) ensures that this agent strictly prefers to wait at least until the arrival of such communication step  $L_i$ , given other agents never exit. Therefore, agent  $i$  is guaranteed to obtain a large number of signals from finite communication steps, if other agents never exit. Note also that the definition of a socially informed agent does not require knowledge of any specific equilibrium. It only depends on the topological structure of the graph and on the parameters in the information exchange game. With the help of socially informed agents, we bypass equilibrium and state the following sufficient condition for perfect learning.

**Proposition 3** *The  $\delta$ -perfect learning occurs in a society  $\{G_n\}_{n=1}^\infty$  under any equilibrium  $\{\sigma^{n,*}\}_{n=1}^\infty$  if*

$$\lim_{n \rightarrow \infty} \frac{1}{n} |SI^n| = 1.$$

Proposition 3 is interesting because we can determine the occurrence of perfect learning through knowledge on the formation of society alone. Given the tractable conditions for socially informed agents, we can check whether a given society sufficiently supports perfect learning under any equilibrium. Especially, given the difficulty of explicitly solving for equilibria of the information exchange game in general cases, Proposition 3 is of more value.



## 4.2 Learning Rates

In this subsection we define the learning rate for  $\delta$ -perfect learning. It is natural to expect similar concepts of learning rates for  $\varepsilon$ -perfect learning and  $\bar{\varepsilon}$ -perfect learning.

**Definition 7** *If  $\delta$ -perfect learning occurs in  $\{G_n\}_{n=1}^\infty$  under equilibria  $\{\sigma^{n,*}\}_{n=1}^\infty$ , then we call the corresponding sequence of tolerances  $\{\delta_n\}_{n=1}^\infty$  the learning rate.*

It is worth highlighting the difference between our learning rate concept and the speed of convergence to a pre-defined consensus mainly employed in observational learning problems. The latter concerns about the time towards a consensus in a circumstance where people make repeated decisions and learn from others' previous decisions to help make their own future decisions (Golub and Jackson (2012a,b,c)). In observational learning problems with repeated decisions, the observable sequence of aggregate decisions naturally reveals the time dynamics of information aggregation. However in direct communication setup, dynamics of information aggregation along the time dimension is largely unobservable, which calls for alternative dimensions to look into the information dynamics. Tolerance parameters  $\{\delta_n\}_{n=1}^\infty$  offer a natural standpoint to look into the information aggregation dynamics. This feature also distinguishes our work from existing literature on learning rate of similar spirit. For example, Acemoglu et al. (2009) attempt to define and investigate an asymptotic learning based rate in an observational learning context. Although their concept also captures a sequence of diminishing probabilities, it does not characterize the learning status in every social network along the society.

On the other hand, it is not trivial to construct concretely the smallest sequence  $\{\delta_n\}_{n=1}^\infty$  for perfect learning, while keeping other parameters fixed. Recall that the sufficient condition part of Proposition 1 implies  $\delta$ -perfect learning occurs with rates  $\{\delta_n\}_{n=1}^\infty$  if

$$\frac{1}{n} \sum_{i=1}^n \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right) \geq 1 - \delta_n \bar{\varepsilon}.$$

Theoretically, if we can directly solve the inequalities with respect to  $\delta_n$ , the achievable fastest learning rate  $\{\delta_n\}_{n=1}^\infty$  is constructed. However, some technical problems prevent us from directly doing so. First, we cannot read off a transparent rate out of the error function. Second, without specific knowledge of network formations, the relation between  $k_i^{n,\sigma^*}$  and  $n$  is hard to be pinned down generally. Moreover, as we will see in the binomial tree example,  $k_i^{n,\sigma^*}$  could be drastically different even for a same graph. Hence, we will first discuss learning rates on specific examples, and generalize to more general categories when possible.

**Example 1 (Isolated Agents)** *When all agents are isolated from each other in a communication network  $G_n$ , we have  $k_i^{n,\sigma^*} = 1$  for every agent  $i$ .*

In Example 1, the negative condition (3.1) is reduced to

$$\operatorname{erf}\left(\varepsilon\sqrt{\frac{\rho+\bar{\rho}}{2}}\right) < (1-\bar{\varepsilon})(1-\delta_n).$$

If parameters are such that  $\operatorname{erf}\left(\varepsilon\sqrt{\frac{\rho+\bar{\rho}}{2}}\right) < (1-\bar{\varepsilon})$ , the above inequality holds for large  $n$  for any vanishing sequence  $\{\delta_n\}_{n=1}^\infty$ . This tells us that in fairly general circumstances, purely isolated society cannot achieve  $\delta$ -perfect learning.

**Example 2 (Complete Graph)** *When the communication network  $G_n$  is a complete graph, and the benefit of getting  $n-1$  new signals justifies the discount of one communication step,  $k_i^{n,\sigma^*} = n$  for every agent  $i$ .*

In Example 2, we have

$$\operatorname{erf}\left(\varepsilon\sqrt{\frac{\rho+\bar{\rho}n}{2}}\right) \geq 1 - \delta_n\bar{\varepsilon}, \quad \forall n \in \mathbb{N},$$

as a sufficient condition for  $\delta$ -perfect learning, which translates to

$$\delta_n \geq \frac{1}{\bar{\varepsilon}} \left( 1 - \operatorname{erf}\left(\varepsilon\sqrt{\frac{\rho+\bar{\rho}n}{2}}\right) \right). \quad (4.4)$$

The sequence of the right hand sides of inequality (4.4) can serve as the learning rate. At the cost of getting a conservative estimate, we approximate the error function in order to get a more transparent learning rate. Note that the error function  $\text{erf}$  can be approximated by

$$1 - \text{erf}(x) < \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}.$$

Therefore a sufficient condition for  $\delta$ -perfect learning is

$$\delta_n \geq \frac{1}{\sqrt{\pi}\bar{\varepsilon}} \frac{1}{\varepsilon\sqrt{\rho + \bar{\rho}n}} \exp\left(-\frac{\varepsilon^2(\rho + \bar{\rho}n)}{4}\right).$$

Keep other parameters fixed, and focus on the relations between population size  $n$  and  $\delta_n$ . We see that  $\delta_n$  could decrease in the order of  $\exp(-\bar{\rho}\varepsilon^2 n/5)$ . This implies that when population grows, the probability that at least  $\bar{\varepsilon}$  fraction of people make the wrong decision decreases very quickly to zero.

Following the idea of error function approximations, we go beyond Example 2 to consider a more general case in which  $k_i^{n,\sigma^*} \geq f(n)$  for every agent  $i$  where  $f(n)$  is a deterministic sequence. A sufficient condition for  $\delta$ -perfect learning is then

$$\delta_n \geq \frac{1}{\sqrt{\pi}\bar{\varepsilon}} \frac{1}{\varepsilon\sqrt{\rho + \bar{\rho}f(n)}} \exp\left(-\frac{\varepsilon^2(\rho + \bar{\rho}f(n))}{4}\right). \quad (4.5)$$

If  $f(n)$  diverges to infinity as  $n$  goes to infinity, the right hand side of inequality (4.5) converges to 0. Keeping other parameters fixed, this implies  $\delta_n$  could decrease in the order of  $\exp(-\bar{\rho}\varepsilon^2 f(n)/5)$ . Formally, we summarize these discussions with the next proposition.

**Proposition 4** *Suppose there exists a diverging sequence  $f(n)$  such that  $k_i^{n,\sigma^*} \geq f(n)$  for any agent  $i$  in network  $G_n$  with associated equilibrium  $\sigma^{n,*}$ , then  $\delta$ -perfect learning could occur with learning rate  $\{\delta_n\}_{n=1}^\infty$ , where each  $\delta_n$  is in the order of  $\exp(-\bar{\rho}\varepsilon^2 f(n)/5)$ .*

The next example is a direct application of Proposition 4.

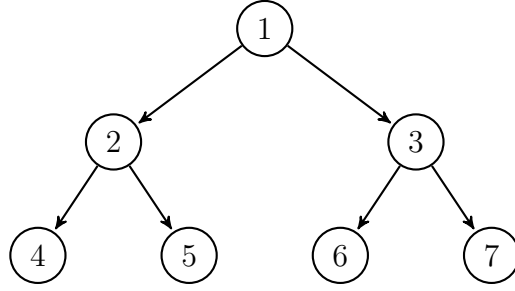
**Example 3** *Suppose  $f(n) = C \cdot n$  where  $0 < C < 1$ , then  $\delta$ -perfect learning could occur*

with learning rate  $\{\delta_n\}_{n=1}^\infty$ , where each  $\delta_n$  is in the order of  $\exp(-\bar{\rho}\varepsilon^2 Cn/5)$ .

An interpretation of this example is that, even if communication is sparse in the sense that each of the agents can only get a small proportion of information in the entire population, perfect learning can still be reached at a fast rate. This example represents a scenario in which communication networks in a society consist of dispersed social groups while agents within these social groups are very closely connected. This is related to interesting results pertaining to social cliques or homophily as discussed in [Golub and Jackson \(2012a,b,c\)](#).

In most models, however, there is no universal bound for  $k_i^{n,\sigma^*}$ . Lemma 3 in the supplementary materials renders Proposition 4 as a special case, but it still does not cover cases when direct conservative estimate for  $k_i^{n,\sigma^*}$  is not feasible. Next we consider such a case: the binomial tree, which is widely considered as an axiomatic representation of various hierarchical structures in the human society [Jackson \(2010\)](#). In particular, as the information flow within a binomial tree can be either from the root to the leafs or from the leafs to the root, binomial trees can accommodate both the top-down and the bottom-up cases of information transmission in various real-world scenarios. Hence, it is instructive to analyze the binomial tree with a few different settings, where we generalize our game by allowing the information sensitiveness  $\psi = \psi_n$  to vary along the society  $\{G_n\}_{n=1}^\infty$ .

**Example 4 (Binomial Tree: Information Flow from Root to Leafs)** *The agents in the communication network  $G_n$  form a binomial tree, where information can only flow from root to leafs. For simplicity, consider only the number of agents  $n$  such that  $n = 1 + 2 + 4 + \dots + 2^{(m_n-1)}$ , where  $m_n$  is the number of layers in the binomial tree. The following graph illustrates such a binomial tree with three layers.*



We will study two scenarios of this binomial tree, in both of which  $\lambda = r$  so that  $\bar{r} = 1/2$ .

- i)  $\psi_n = \rho = \bar{\rho} = 1$ . For agent 1 on the top layer, he should exit right after step 0 because he does not have any chance to receive others' private information. For agent 2 and 3, who are on the second top layer, they decide between step 0 and 1. A simple calculation on their pay off functions reveals that they should exit after step 0. Agents 4, 5, 6, 7 who are on the third layer potentially should decide between 0,1 and 2 steps. But since agents 2 and 3 cannot not pass through agent 1's info, step 2 is eliminated before any calculation. So agents on the third layer actually faces same decision problems as agents on the second layer. Continue with the same argument till the  $m_n$ 'th layer, we learn that everyone in the communication network exits right after she gets the private signal. Therefore, this scenario is the same as *isolated points* in terms of information aggregation.

In general, as depicted in this subcase i), when the communication game is less information sensitive, the precision of the prior is higher, or the precision of the private signal is lower, it is less likely to achieve  $\delta$ -perfect learning, even if the agents are well connected.

- ii)  $\psi_n < \frac{2}{\rho+(m_n-1)\bar{\rho}} - \frac{1}{\rho+m_n\bar{\rho}}$  and  $\varepsilon^2 < -\frac{4}{\bar{\rho}} \log \left( \frac{1}{2} \sqrt{\frac{\rho+2\bar{\rho}}{\rho+\bar{\rho}}} \right)$ . Same as subcase i), agent 1 does not have a choice. For agents on the second layer to choose exit at step 1, we need  $\psi_n < \frac{2}{\rho+\bar{\rho}} - \frac{1}{\rho+2\bar{\rho}}$ . For agents on the third layer to exit at step 2, we need

$$\psi_n < \min \left\{ \frac{2}{\rho+\bar{\rho}} - \frac{1}{\rho+2\bar{\rho}}, \frac{2}{\rho+2\bar{\rho}} - \frac{1}{\rho+3\bar{\rho}} \right\} = \frac{2}{\rho+2\bar{\rho}} - \frac{1}{\rho+3\bar{\rho}}.$$

In general, an agent on layer  $j$  wait till the  $j - 1$  step if

$$\psi_n < \min \{g(1), \dots, g(j-1)\} = g(j-1).$$

where  $g(x) = \frac{2}{\rho+x\bar{\rho}} - \frac{1}{\rho+(x+1)\bar{\rho}}$ . The last equality holds because  $g(x)$  is a decreasing function, thanks to  $g'(x) < 0$ . Hence under equilibrium, agents on layer  $j$  have  $j$  signals. In particular, agents in the last layer each has  $m_n = \log_2(n+1)$  signals. Note that there are  $\frac{n+1}{2}$  agents in this layer. Using (3.2), a similar derivation to that in Example 2 leads to that the learning rate  $\delta_n$  should be

$$\delta_n \geq \frac{1}{n\varepsilon\bar{\varepsilon}\sqrt{\pi}} \sum_{i=1}^{\log_2(n+1)} 2^{j-1} \frac{1}{\sqrt{\rho + \bar{\rho}j}} \exp\left(-\frac{\varepsilon^2(\rho + \bar{\rho}j)}{4}\right).$$

To unravel right hand side of the above inequality, we let

$$h(x) = 2^{x-1} \frac{1}{\sqrt{\rho + \bar{\rho}x}} \exp\left(-\frac{\varepsilon^2(\rho + \bar{\rho}x)}{4}\right).$$

Then  $h(x)$  is monotone increasing, because  $h(x+1)/h(x) > 1$  under our condition.

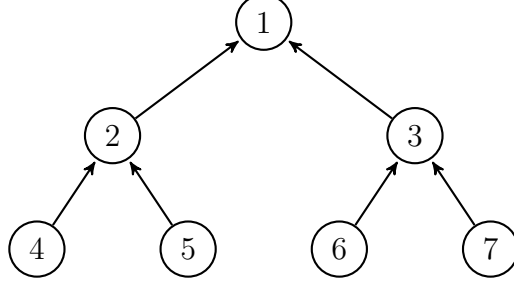
Therefore, it is sufficient to have

$$\delta_n \geq \frac{1}{n\varepsilon\bar{\varepsilon}\sqrt{\pi}} \log_2(n+1) \cdot 2^{\log_2(n+1)-1} \frac{1}{\sqrt{\rho + \bar{\rho}\log_2(n+1)}} \exp\left(-\frac{\varepsilon^2(\rho + \bar{\rho}\log_2(n+1))}{4}\right).$$

Therefore  $\delta_n$  could decay in the order of  $\sqrt{\log(n+1)} \cdot (n+1)^{-\varepsilon^2\bar{\rho}/4}$ , which is a polynomial rate.

Compared to the complete graph, the binomial tree aggregates information much slower. The difference in learning rates arises not only from the physical network structures, but also from different strategic interactions among agents in the two environments. Next, we consider a twin case of the binomial tree, in which information flows in the opposite direction.

**Example 5 (Binomial Tree: Information Flow from Leafs to Root)** Now let information flow from leafs to root, i.e., reverse all the directed edges in Example 4. The following graph illustrates such a binomial tree with three layers.



We give the following results. The detailed analysis is similar to Example 4.

- i)  $\psi_n = \rho = \bar{\rho} = 1$ . All agents exit after time 0.
- ii)  $\psi_n < \frac{2}{\rho + 2^{(m_n-1)}\bar{\rho}} - \frac{1}{\rho + 2^{m_n}\bar{\rho}}$ . All agents get the maximum number of signals that they could possibly get, then  $\delta_n$  can be such that

$$\delta_n \geq \frac{1}{n\varepsilon\bar{\varepsilon}\sqrt{\pi}} \sum_{j=1}^{\log_2(n+1)} 2^{j-1} \frac{1}{\sqrt{\rho + 2^{(m_n-j+1)}\bar{\rho}}} \exp\left(-\frac{\varepsilon^2(\rho + 2^{(m_n-j+1)}\bar{\rho})}{4}\right).$$

A conservative estimate on the summation on the right hand side would give  $\delta_n \sim n^{-3/4}$ , a much faster rate than that in Example 4 when  $\varepsilon^2\bar{\rho} \ll 3$ , which can be considered as a typical case as we have in mind very small  $\varepsilon$ .

Note that different information flow directions matter for learning rates. When parameters are in comparable range, the bottom-up case exhibits a higher learning rate than the top-down case does. In other words, the bottom-up organization of information flow within a binomial tree is more efficient in terms of aggregating information. This result is consistent with early economics and sociology literature; Hayek (1945) for example, highlight the importance of dispersed information sources.

The four sub-cases under binomial tree setting demonstrate that beyond a directed graphical model, contextual information is also very important for information aggregation. These comparisons are made possible only with help of our concept of finite

population learning and  $\delta$ -perfect learning. Properties or statistics of the graph alone cannot determine the learning status. Rather, as we argued from the very beginning of this work, the enriched game theory plus graphical modeling approach are both interesting and necessary in helping understand the information aggregation in social networks.

## 5 Remarks and Further Research

We have proposed a finite population learning concept to capture the level of information aggregation in any given communication network. In our framework, one equilibrium outcome, i.e., the number of signals obtained by an agent when she makes a decision, plays a key role. This equilibrium outcome is computable ([McKelvey and McLennan \(1996\)](#)), which also allows us to numerically check the learning status. Different from existing literature that mainly addresses the learning behavior at the limit, this new concept helps reveal explicit interplays among time discount, frequency of communication, information precision and information sensitiveness of the decision problem in any finite communication network. As the total amount of information is fixed in a given finite network, our approach enables meaningful comparative statics regarding the effectiveness of information aggregation in networks. We also provide conditions for learning under a particular equilibrium, under any equilibrium, and under all equilibria, respectively. Thanks to its tractability and transparency, the finite population learning concept offers a solid foundation to investigate long run dynamics of learning behavior and the associated learning rates as population diverges.

Our analysis is also subject to certain limitations, which suggest directions for future research. In our model, complete information on the structure of a given communication network is required in determining the number of signals obtained by an agent, and in checking its corresponding learning status. In some circumstances, researchers do not want to assume such specific information; rather they want to get some understanding of the learning status regarding a large class of networks. This goal calls for some new



criteria that can determine the learning status for given classes of networks with given finite population; ideally, these criteria should be formulated in terms of some summary statistics of these networks. Relaxing the knowledge on specific network structure may also lead to more general results about the learning rates. However, this task is challenging within the current finite population learning framework. Specifically, our established conditions for finite population learning require all agents' exact numbers of signals upon their exits. Only knowing some commonly used summary statistics of the associated graphs can hardly help offer good estimates of these numbers of signals, mostly because these numbers of signals are also affected by other parameters not directly related to the network structure, such as the information precisions and the tolerances of learning. Therefore, the learning status of a certain class of communication networks is largely undetermined if we just consider properties of the graphs. To address this issue, we would like to have novel statistical properties of communication networks that are more friendly to the analysis of communication learning. [Golub and Jackson \(2012a,b,c\)](#) are promising attempts towards this direction, but their results are limited to the context of non-Bayesian observational learning.

Another line of generalization is to make our model more flexible and realistic. For example, the current setting assumes that agents have private signals with the same precision, which amounts to assuming that the total amount of information grows linearly with the population size when we consider the perfect learning. It might be interesting to allow the total information to increase in a nonlinear (e.g., log rate) fashion with the population size, and allow non-uniform distribution of signal precisions among agents. Also, even when we focus on certain classes of networks without specifying complete network structure, it is still assumed that any agent in the communication network knows the complete network structure. This assumption can be relaxed by limiting agents' knowledge on the network to a certain local neighborhood, and infer other parts of the network according to her local knowledge. Keeping the Bayesian learning paradigm, other potential generalizations of our current work include considering the implications of

correlated private information among agents, and heterogeneous characteristics of agents.

## References

- ACEMOGLU, D., BIMPIKIS, K. and OZDAGLAR, A. (2012a). Dynamics of information exchange in endogenous social networks. *Theoretical Economics*.
- ACEMOGLU, D., CARVALHO, V., OZDAGLAR, A. and TAHBAZ-SALEHI, A. (2012b). The network origins of aggregate fluctuations. *Econometrica*.
- ACEMOGLU, D., DAHLEH, M., LOBEL, I. and OZDAGLAR, A. (2009). Rate of convergence of learning in social networks. *Proceedings of the American Control Conference*.
- ACEMOGLU, D., DAHLEH, M., LOBEL, I. and OZDAGLAR, A. (2011). Bayesian learning in social networks. *Review of Economic Studies*, **78** 1201–1236.
- ACEMOGLU, D. and OZDAGLAR, A. (2010). Opinion dynamics and learning in social networks. *Dynamic Games and Applications*, **1** 3–49.
- BANERJEE, A. (1992). A simple model of herd behavior. *Quarterly Journal of Economics*, **107** 797–817.
- BANERJEE, A. and FUDENBERG, D. (2004). Word-of-mouth learning. *Games and Economic Behavior*, **46** 1–2.
- BIKHCHANDANI, S., HIRSHLEIFER, D. and WELCH, I. (1992). A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of Political Economy*, **100** 992–1026.
- CRAWFORD, V. and SOBEL, J. (1982). Strategic information transmission. *Econometrica*, **50** 1431–1451.
- DEGROOT, M. (1974). Reaching a consensus. *Journal of the American Statistical Association*, **69** 118–121.
- DEGROOT, M. (2003). Persuasion bias, social influence, and unidimensional opinions. *Quarterly Journal of Economics*, **118** 909–968.

- GALEOTTI, A., GHIGLINO, C. and SQUINTANI, F. (2011). Strategic information transmission in networks. *working paper*.
- GOLUB, B. and JACKSON, M. (2010). Naive learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics*, **2** 112–149.
- GOLUB, B. and JACKSON, M. (2012a). Does homophily predict consensus times? testing a model of network structure via a dynamic process. *Review of Network Economics*.
- GOLUB, B. and JACKSON, M. (2012b). How homophily affects the speed of learning and best-response dynamics. *Quarterly Journal of Economics*.
- GOLUB, B. and JACKSON, M. (2012c). Network structure and the speed of learning: Measuring homophily based on its consequences. *Annals of Economics and Statistics*.
- GOYAL, S. (2009). *Connections: An Introduction to the Economics of Networks*. Princeton University Press.
- HAGENBACH, J. and KOESSLER, F. (2010). Strategic communication networks. *Review of Economic Studies*, **77** 1072–1099.
- HAYEK, F. (1945). The use of knowledge in society. *The American Economic Review*, **35** 519–530.
- JACKSON, M. (2010). *An Overview of Social Networks and Economic Applications*. North Holland.
- KOLACAZYK, E. (2009). *Statistical Analysis of Network Data: Methods and Models*. Springer.
- MCKELVEY, R. and MCLENNAN, A. (1996). *Computation of Equilibria in Finite Games*. Elsevier.
- MOBIUS, M., PHAN, T. and SZEIDL, A. (2010). Treasure hunt. *working paper*.

- MULLER-FRANK, M. (2012). A general framework for rational learning in social networks.  
*Theoretical Economics*.
- NEWMAN, M. (2010). *Networks: an Introduction*. Oxford University Press.
- SMITH, L. and SORENSEN, P. (2000). Pathological outcomes of observational learning.  
*Econometrica*, **68** 371–398.
- SMITH, L. and SORENSEN, P. (2008). Rational social learning with random sampling.  
*working paper*.
- TOPKIS, D. (1979). Equilibrium points in non-zero sum n-person submodular games.  
*SIAM Journal of Control and Optimization*, **17** 773–787.

## Supplementary Materials

The supplementary materials provide proofs and generalized results of corresponding parts in the main text.

**Proof of Proposition 1.** To prevent  $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning, it is enough to show that a lower bound of  $\mathbb{P}_{\sigma^{n,*}} \left( \frac{1}{n} \sum_{i=1}^n (1 - M_i^{n,\varepsilon}) \geq \bar{\varepsilon} \right)$  is greater than  $\delta$ . It follows from Markov inequality,

$$\mathbb{P}_{\sigma^{n,*}} \left( \frac{1}{n} \sum_{i=1}^n M_i^{n,\varepsilon} > 1 - \bar{\varepsilon} \right) \leq n^{-1}(1 - \bar{\varepsilon})^{-1} \sum_{i=1}^n \mathbb{E}_{\sigma^{n,*}} M_i^{n,\varepsilon} = n^{-1}(1 - \bar{\varepsilon})^{-1} \sum_{i=1}^n \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right).$$

This implies that

$$\mathbb{P}_{\sigma^{n,*}} \left( \frac{1}{n} \sum_{i=1}^n (1 - M_i^{n,\varepsilon}) \geq \bar{\varepsilon} \right) > 1 - n^{-1}(1 - \bar{\varepsilon})^{-1} \sum_{i=1}^n \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right).$$

Therefore, it is enough to take

$$1 - n^{-1}(1 - \bar{\varepsilon})^{-1} \sum_{i=1}^n \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right) > \delta,$$

which concludes that condition (3.1) is necessary for  $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning.

To ensure  $(\varepsilon, \bar{\varepsilon}, \delta)$ -learning, note that

$$\mathbb{P}_{\sigma^{n,*}} \left( \frac{1}{n} \sum_{i=1}^n (1 - M_i^{n,\varepsilon}) \geq \bar{\varepsilon} \right) \leq \frac{\mathbb{E}_{\sigma^{n,*}} (\sum_{i=1}^n (1 - M_i^{n,\varepsilon}))}{n\bar{\varepsilon}} = \frac{n - \sum_{i=1}^n \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n,\sigma^*}}{2}} \right)}{n\bar{\varepsilon}}.$$

Demanding the right hand side of the above inequality no larger than  $\delta$ , is the same as assuming condition (3.2). This completes the proof.

The following provides a more general sufficient condition for  $\delta$ -perfect learning. Given equilibria  $\{\sigma^{n,*}\}_{n=1}^\infty$ , let  $f_1 \geq f_2 \geq \dots \geq f_J$ , where each  $f_j(n)$  is a monotone increasing

function (not necessarily strictly increasing) on  $n$ , and let  $\{b_n^j, j = 1, \dots, J\}$  be such that

$$\frac{|\{i : k_i^{n, \sigma^*} \geq f_1(n)\}|}{n} \geq b_n^1,$$

$$\frac{|\{i : f_1(n) > k_i^{n, \sigma^*} \geq f_2(n)\}|}{n} \geq b_n^2,$$

and up until

$$\frac{|\{i : f_{J-1}(n) > k_i^{n, \sigma^*} \geq f_J(n)\}|}{n} \geq b_n^J.$$

Clearly,  $b_n^1, \dots, b_n^J \in (0, 1)$  and  $0 \leq b_n^1 + \dots + b_n^J \leq 1$ . The rest agents  $i$ 's are such that  $f_J(n) > k_i^{n, \sigma^*} \geq 1$ . Their fraction is at most  $1 - (b_n^1 + \dots + b_n^J)$ .

**LEMMA 3**  *$\delta$ -perfect learning occurs if*

$$(a) \lim_{n \rightarrow \infty} \sum_{j=1}^J b_n^j = 1,$$

$$(b) \text{ for each } j \in \{1, \dots, J\}, \lim_{n \rightarrow \infty} b_n^j \left(1 - \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} f_j(n)}{2}} \right)\right) = 0.$$

**Proof of Lemma 3.** Recall that a sufficient condition for  $(\varepsilon, \bar{\varepsilon}, \delta_n)$ -learning is

$$\frac{1}{n} \sum_{i=1}^n \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} k_i^{n, \sigma^*}}{2}} \right) \geq 1 - \delta_n \varepsilon.$$

Then by the definition of  $b_n^j$  and  $f_j$ , it is enough to have

$$\sum_{j=1}^J \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} f_j(n)}{2}} \right) \cdot b_n^j + \left(1 - \sum_{j=1}^J b_n^j\right) \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho}}{2}} \right) \geq 1 - \delta_n \varepsilon.$$

This translates to

$$\delta_n \geq \frac{1}{\bar{\varepsilon}} \left( \sum_{j=1}^J b_n^j \left(1 - \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} f_j(n)}{2}} \right)\right) + \left(1 - \sum_{j=1}^J b_n^j\right) \left(1 - \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho}}{2}} \right)\right) \right).$$

To ensure the existence of  $\{\delta_n\}$  such that  $\lim_{n \rightarrow \infty} \delta_n = 0$ , it is enough to have

$\lim_{n \rightarrow \infty} \sum_{j=1}^J b_n^j = 1$  and

$$\lim_{n \rightarrow \infty} b_n^j \left( 1 - \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} f_j(n)}{2}} \right) \right) = 0, \quad \text{for } j \leq J.$$

Note that if  $f_j$  does not increase strictly for  $n \geq N^*$ ,  $b_n^j$  needs to decrease to 0. Also, allowing more than one tolerances among  $\varepsilon$ ,  $\bar{\varepsilon}$ ,  $\delta$  to vary with population size  $n$  leads to interesting learning results. In particular, from the proof of Lemma 3, a sufficient condition for  $(\varepsilon, \bar{\varepsilon}_n, \delta_n)$ - learning is

$$\delta_n \bar{\varepsilon}_n \geq \sum_{j=1}^J b_n^j \left( 1 - \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho} f_j(n)}{2}} \right) \right) + \left( 1 - \sum_{j=1}^J b_n^j \right) \left( 1 - \operatorname{erf} \left( \varepsilon \sqrt{\frac{\rho + \bar{\rho}}{2}} \right) \right).$$

In this condition, the role of  $\delta_n$  and that of  $\bar{\varepsilon}_n$  are completely interchangeable, which implies that we can trade in some probabilistic confidence for some fraction of agents who make wrong decisions.

The following provides a generalized version of Lemma 2 when Assumption 2 is relaxed.

**LEMMA 4 (Generalized Lemma 2)** *For any agent  $i$ , either the communication steps she optimally experiences before taking an action in any social network  $G_n$  along a society  $\{G_n\}_{n=1}^\infty$  is bounded from above by a constant independent of  $n$ , or she waits until the maximum allowable step. Specifically,*

(a) *If  $(\rho + \bar{\rho})\psi > 1$ , then for any agent  $i$*

$$l_i^{n, \sigma^*} \leq l_i^n < \min \left\{ (L_i^n)_{\max}, \ln \left( 1 - \frac{1}{(\rho + \bar{\rho})\psi} \right) / \ln \bar{r} \right\},$$

*where  $l_i^n$  stands for agent  $i$ 's optimal communication rounds given that other agents wait till the maximum allowable step.*

(b) *If  $(\rho + \bar{\rho})\psi \leq 0$  (equivalently,  $\psi \leq 0$ ), then for any agent  $i$*

$$l_i^{n, \sigma^*} = l_i^n = (L_i^n)_{\max}.$$



(c) If  $0 < (\rho + \bar{\rho})\psi \leq 1$ , then there are two subcases.

(c.1) For agent  $i$  with

$$\lim_{n \rightarrow \infty} |B_i^n| < \frac{1 - \rho\psi}{\bar{\rho}\psi},$$

where  $B_i^n$  is the set of agents whose signals agent  $i$  can get if no one exits before maximum allowable step, we have

$$l_i^{n,\sigma^*} = l_i^n = (L_i^n)_{\max}.$$

(c.2) For agent  $i$  with

$$\lim_{n \rightarrow \infty} |B_i^n| \geq \frac{1 - \rho\psi}{\bar{\rho}\psi},$$

we have either

$$l_i^{n,\sigma^*} \leq l_i^n \leq \min \left( (L_i^n)_{\max}, l_i^{\{G_n\}_{n=1}^\infty} \right),$$

or

$$l_i^{n,\sigma^*} = (L_i^n)_{\max},$$

where  $l_i^{\{G_n\}_{n=1}^\infty}$  is a constant that depends on the society and agent  $i$ 's position in the society, but does not change with  $n$ .

**Proof of Lemma 4.** We proceed case by case.

CASE (a),  $(\rho + \bar{\rho})\psi > 1$ .

In this case, agent  $i$  enjoys a positive payoff  $\psi - \frac{1}{\rho + \bar{\rho}}$  if she exists at  $t = 0$  and does not communicate with anyone else. Note that her expected payoff by taking  $l_i^n$  communication steps is strictly upper bounded by  $\bar{r}^{l_i^n} \psi$ . Therefore, it is suboptimal for her to choose a  $l_i^n$  such that

$$\bar{r}^{l_i^n} \psi \leq \psi - \frac{1}{\rho + \bar{\rho}},$$

which implies

$$l_i^n < \ln \left( 1 - \frac{1}{(\rho + \bar{\rho})\psi} \right) / \ln \bar{r}$$

is necessary for agent  $i$ 's optimality. It is obvious that  $l_i^{n,\sigma^*} \leq l_i^n$ , since other agents do not necessarily wait forever in an equilibrium, so that it may be optimal for agent  $i$  to exit earlier too. We get the result by combining these with the upper bound  $l_i^n \leq (L_i^n)_{max}$ .

CASE (b),  $(\rho + \bar{\rho})\psi \leq 0$ .

Now agent  $i$  always gets a negative payoff whenever she exits. Because waiting discounts the negative payoff, she optimally chooses to wait as long as possible, no matter what other agents do. Therefore,  $l_i^{n,\sigma^*} = l_i^n = (L_i^n)_{max}$ .

CASE (c.1),  $0 < (\rho + \bar{\rho})\psi \leq 1$  and  $\lim_{n \rightarrow \infty} |B_i^n| < \frac{1-\rho\psi}{\bar{\rho}\psi}$ .

The maximum number of private signals agent  $i$  can get is  $|B_i^n|$ . Again, agent  $i$  always gets a negative payoff whenever she exists. Hence,  $l_i^{n,\sigma^*} = l_i^n = (L_i^n)_{max}$ .

CASE (c.2),  $0 < (\rho + \bar{\rho})\psi \leq 1$  and  $\lim_{n \rightarrow \infty} |B_i^n| \geq \frac{1-\rho\psi}{\bar{\rho}\psi}$ .

For any  $G_n$  with  $|B_i^n| \geq \frac{1-\rho\psi}{\bar{\rho}\psi}$ , we consider the communication step  $(L_i^n)_{max}$  when agent  $i$  obtains signals from all her sources  $B_i^n$ , provided others wait maximum steps. Note that  $(L_i^n)_{max}$  is non-decreasing in  $n$  for any agent  $i$  (by the no deleting assumption), and  $|B_{i,l}^n|$  is strictly monotone increasing in  $l$  when  $l \leq (L_i^n)_{max}$ .

Also for a given communication network  $G_n$ , there exists one communication step  $l_i^{n'}$  such that after this step agent  $i$  gets positive payoff, given that other agents wait maximum steps. Hence, it is suboptimal for her to wait longer than  $l_i^{n'}$  if

$$\bar{r}\psi \leq \psi - \frac{1}{\rho + \bar{\rho}|B_{i,l_i^{n'}}^n|},$$

which implies

$$|B_{i,l_i^n}^n| < \frac{\lambda + r - \rho r \psi}{\bar{\rho} r \psi} \quad (5.1)$$

is necessary for agent  $l_i^{n'}$ 's optimality.

Now we consider two sub-cases. First is when  $\lim_{n \rightarrow \infty} |B_i^n| < \infty$ . Then we must have  $\lim_{n \rightarrow \infty} (L_i^n)_{max} < \infty$ , since  $(L_i^n)_{max} \leq |B_i^n|$ . Hence,

$$l_i^n \leq \lim_{n \rightarrow \infty} (L_i^n)_{max} < \infty,$$

for all  $G_n$  satisfying  $|B_i^n| \geq \frac{1-\rho\psi}{\bar{\rho}\psi}$  and  $\lim_{n \rightarrow \infty} |B_i^n| < \infty$ . We denote  $\lim_{n \rightarrow \infty} (L_i^n)_{\max}$  as  $l_{1i}^{\{G_n\}_{n=1}^\infty}$ , which is a constant that depends on the society and agent  $i$ 's position in the society and does not change with respect to  $n$ .

Second, we discuss the case when  $\lim_{n \rightarrow \infty} |B_i^n| = \infty$ . Now there should be either  $\lim_{n \rightarrow \infty} (L_i^n)_{\max} < \infty$  or  $\lim_{n \rightarrow \infty} (L_i^n)_{\max} = \infty$ . In the former scenario, we have  $l_i^n \leq \lim_{n \rightarrow \infty} (L_i^n)_{\max} = l_{1i}^{\{G_n\}_{n=1}^\infty}$  for all  $G_n$ . In the latter case, as  $(L_i^n)_{\max}$  is non-decreasing in  $n$  for any given  $i$  and  $|B_{i,l}^n|$  is strictly monotone increasing in  $l$  when  $l \leq (L_i^n)_{\max}$  for any  $G_n$ , there exists a largest  $G_N$  with its associated  $(L_i^N)_{\max}$  that satisfies condition (5.1). Hence, by (5.1) we obtain

$$l_i^n \leq (L_i^N)_{\max},$$

for all  $G_n$  satisfying  $|B_i^n| \geq \frac{1-\rho\psi}{\bar{\rho}\psi}$ ,  $\lim_{n \rightarrow \infty} |B_i^n| = \infty$  and  $\lim_{n \rightarrow \infty} (L_i^n)_{\max} = \infty$ . We denote such  $(L_i^N)_{\max}$  as  $l_{2i}^{\{G_n\}_{n=1}^\infty}$ , which is again a constant that depends on the society and agent  $i$ 's position in the society and does not change with respect to  $n$ . To sum up, we denote by  $l_i^{\{G_n\}_{n=1}^\infty}$  either  $l_{1i}^{\{G_n\}_{n=1}^\infty}$  or  $l_{2i}^{\{G_n\}_{n=1}^\infty}$  in respective cases, and it follows  $l_i^n \leq l_i^{\{G_n\}_{n=1}^\infty}$  for agent  $i$  in such  $G_n$  with  $|B_i^n| \geq \frac{1-\rho\psi}{\bar{\rho}\psi}$ , where  $l_i^{\{G_n\}_{n=1}^\infty}$  is independent of  $n$ .

As for  $l_i^{n,\sigma^*}$ , since other agents play equilibrium strategies, agent  $i$  gets weakly fewer signals than that she can get when other agents wait maximum steps. There can be two cases, either she gets positive payoff and takes an action weakly earlier, namely,  $l_i^{n,\sigma^*} \leq l_i^n$ , or she cannot get enough signals to ensure a positive payoff so that she optimally until the maximum allowable step, i.e.,  $l_i^{n,\sigma^*} = (L_i^n)_{\max}$ . This concludes the proof.

**Proof of Proposition 3.** By Lemma 3, it suffices to show that  $\lim_{n \rightarrow \infty} k_i^{n,\sigma^*} = \infty$  under any equilibria  $\{\sigma^{n,*}\}_{n=1}^\infty$  for any socially informed agent  $i$ . In the following, we consider a fixed socially informed agent  $i$ . Recall that in Definition 6,  $L_i$  is defined as the smallest positive integer such that  $\lim_{n \rightarrow \infty} |B_{i,L_i}^n| = \infty$ . Denote by  $B_{i,l}^{n,\sigma^*}$  the set of agents whose signals can reach  $i$  in the first  $l$  rounds of communication under equilibrium  $\sigma^{n,*}$ .

The problem is simple when  $L_i = 1$ . Clearly,  $B_{i,1}^{n,\sigma^*} = B_{i,1}^n$  under any equilibrium  $\sigma^{n,*}$ .

As agent  $i$  is socially informed, we have for sufficiently large  $n$

$$\psi - \frac{1}{\rho + \bar{\rho}|B_{i,1}^{n,\sigma^*}|} > 0 \text{ under any } \sigma^{n,*}$$

and

$$\bar{r} \left( \psi - \frac{1}{\rho + \bar{\rho}|B_{i,1}^{n,\sigma^*}|} \right) > \psi - \frac{1}{\rho + \bar{\rho}} \text{ under any } \sigma^{n,*}.$$

The above display implies that agent  $i$  should at least wait for one communication round. Hence,  $k_i^{n,\sigma^*} \geq |B_{i,1}^{n,\sigma^*}|$  under any  $\sigma^{n,*}$  for sufficiently large  $n$ . As a consequence,  $\lim_{n \rightarrow \infty} k_i^{n,\sigma^*} \geq \lim_{n \rightarrow \infty} |B_{i,1}^{n,\sigma^*}| = \lim_{n \rightarrow \infty} |B_{i,1}^n| = \infty$  under any  $\{\sigma^{n,*}\}_{n=1}^\infty$ .

The following discussion is on the cases when  $L_i \geq 2$ . We proceed through three steps.

STEP 1. We claim when  $L_i \geq 2$ , for sufficiently large  $n$ , there exists at least one path  $\{j_{L_i-1}, j_{L_i-2}, \dots, j_1, i\}$  from  $j_{L_i-1}$  to  $i$  such that

$$\lim_{n \rightarrow \infty} |B_{j_{L_i-l}, l}^n| = \infty \text{ for all } l \in \{1, \dots, L_i - 1\}. \quad (5.2)$$

Now we construct the path  $\{j_{L_i-1}, j_{L_i-2}, \dots, j_1, i\}$  that satisfies condition (5.2). Because  $L_i$  is the smallest integer  $j$  such that  $\lim_{n \rightarrow \infty} |B_{i,j}^n| = \infty$ ,  $B_{i,L_i-1}^n \setminus B_{i,L_i-2}^n$ , the set of agents that are of distance  $L_i - 1$  to  $i$ , must be finite in the limit, i.e.,  $\lim_{n \rightarrow \infty} |B_{i,L_i-1}^n \setminus B_{i,L_i-2}^n| < \infty$ . Therefore, there is at least one agent  $j$  of distance  $L_i - 1$  to  $i$ , such that  $\lim_{n \rightarrow \infty} |B_{j,1}^n| = \infty$ . We denote one of such agents  $j$  as  $j_{L_i-1}$ . If  $L_i = 2$ , the desired path has been constructed. When  $L_i \geq 3$ , choose any path  $\{j_{L_i-1}, j_{L_i-2}, \dots, j_1, i\}$  from the chosen  $j_{L_i-1}$  to  $i$ . Clearly,  $j_{L_i-l} \in B_{i,L_i-l}^n$ . Moreover, condition (5.2) is satisfied in view of  $\lim_{n \rightarrow \infty} |B_{j_{L_i-1},1}^n| = \infty$ .

STEP 2. We next argue that when  $L_i \geq 2$ , agent  $j_{L_i-l}$  on the path  $\{j_{L_i-1}, j_{L_i-2}, \dots, j_1, i\}$  will not exit before she experiences  $l$  communication steps under any equilibrium  $\sigma^{n,*}$  provided that  $n$  is sufficiently large. It is worth noting that agent  $j_{L_i-l}$  does not necessarily get a positive payoff when she experiences  $l$  communication steps in equilibrium.

We will see this by induction from  $j_{L_i-1}$  to  $j_1$  sequentially. We first show that agent  $j_{L_i-1}$  will not exit before she experiences her first communication step in any equilibrium

$\sigma^{n,*}$  provided that  $n$  is sufficiently large. It requires that there exists  $N$  such that for all social networks  $G_n \in \{G_n\}_{n=1}^\infty$  and its associated equilibrium  $\sigma^{n,*}$  with  $n \geq N$ ,

$$\bar{r} \left( \psi - \frac{1}{\rho + \bar{\rho} |B_{j_{L_i-1},1}^{n,\sigma^*}|} \right) > \psi - \frac{1}{\rho + \bar{\rho}}. \quad (5.3)$$

To validate condition (5.3), recall condition (4.3) from Definition 6 for  $l = L_i - 1$ , which states that there exists  $N$  such that for all social networks  $G_n \in \{G_n\}_{n=1}^\infty$  with  $n \geq N$  it holds

$$\bar{r} \left( \psi - \frac{1}{\rho + \bar{\rho} |B_{i,L_i}^n|} \right) > \psi - \frac{1}{\rho + \bar{\rho} |B_{i,L_i-1}^n|}. \quad (5.4)$$

By the definition of  $L_i$ , the construction of  $j_{L_i-1}$  and the fact that  $B_{j_{L_i-1},1}^{n,\sigma^*} = B_{j_{L_i-1},1}^n$  under any equilibrium  $\sigma^{n,*}$  with any  $n$ , we know that  $\lim_{n \rightarrow \infty} |B_{j_{L_i-1},1}^{n,\sigma^*}| = \lim_{n \rightarrow \infty} |B_{j_{L_i-1},1}^n| = \infty$  under any  $\sigma^{n,*}$  and  $\lim_{n \rightarrow \infty} |B_{i,L_i}^n| < \infty$ . Also we have  $|B_{i,L_i-1}^n| \geq 1$ . Note that the right hand side of condition (5.4) is greater than or equal to the right hand side of condition (5.3), we obtain easily that (5.3) holds for sufficiently large  $n$ . Hence we get that agent  $j_{L_i-1}$  will not exit before she experiences her first communication step under any  $\sigma^{n,*}$  provided that  $n$  is sufficiently large.

We then show that agent  $j_{L_i-2}$  (for  $L_i \geq 3$ ) will not exit before she experiences her second communication step under any equilibrium for sufficiently large  $n$ . It requires that there exists  $N$  such that for all social networks  $G_n \in \{G_n\}_{n=1}^\infty$  and its associated equilibrium  $\sigma^{n,*}$  with  $n \geq N$ ,

$$\bar{r}^2 \left( \psi - \frac{1}{\rho + \bar{\rho} |B_{j_{L_i-2},2}^{n,\sigma^*}|} \right) > \psi - \frac{1}{\rho + \bar{\rho}}, \quad (5.5)$$

and

$$\bar{r}^2 \left( \psi - \frac{1}{\rho + \bar{\rho} |B_{j_{L_i-2},2}^{n,\sigma^*}|} \right) > \bar{r} \left( \psi - \frac{1}{\rho + \bar{\rho} |B_{j_{L_i-2},1}^{n,\sigma^*}|} \right). \quad (5.6)$$

To validate (5.5) and (5.6), we use again the condition (4.3) from Definition 6 for  $l = L_i - 2$  and  $l = L_i - 1$ , which state that there exists  $N$  such that for all social networks

$G_n \in \{G_n\}_{n=1}^\infty$  with  $n \geq N$  we have

$$\bar{r}^2 \left( \psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_i}^n|} \right) > \psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_{i-2}}^n|}, \quad (5.7)$$

and

$$\bar{r}^2 \left( \psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_i}^n|} \right) > \bar{r} \left( \psi - \frac{1}{\rho + \bar{\rho}|B_{i,L_{i-1}}^n|} \right). \quad (5.8)$$

Similarly, by the definition of  $L_i$  and the construction of  $j_{L_i-1}$  and  $j_{L_i-2}$ , we know that  $\lim_{n \rightarrow \infty} |B_{j_{L_i-2},2}^n| = \lim_{n \rightarrow \infty} |B_{i,L_i}^n| = \infty$ ,  $\lim_{n \rightarrow \infty} |B_{i,L_{i-1}}^n| < \infty$ ,  $\lim_{n \rightarrow \infty} |B_{i,L_{i-2}}^n| < \infty$ , and  $\lim_{n \rightarrow \infty} |B_{j_{L_i-2},1}^{n,\sigma^*}| \leq \lim_{n \rightarrow \infty} |B_{j_{L_i-2},1}^n| < \infty$  under any equilibrium  $\sigma^{n,*}$ . Also we have  $|B_{i,L_{i-2}}^n| \geq 1$  and  $B_{j_{L_i-2},1}^{n,\sigma^*} \subseteq B_{j_{L_i-2},1}^n \subseteq B_{i,L_{i-1}}^n$  (and thus  $|B_{i,L_{i-1}}^n| \geq |B_{j_{L_i-2},1}^n| \geq |B_{j_{L_i-2},1}^{n,\sigma^*}|$ ) for any  $n$  under any equilibrium  $\sigma^{n,*}$ . Note that the right hand side of condition (5.7) is greater than or equal to the right hand side of condition (5.5), and the right hand side of condition (5.8) is greater than or equal to the right hand side of condition (5.6). Then it can be verified that the next two inequalities hold for sufficiently large  $n$ , the right hand sides of which are the same as those in conditions (5.5) and (5.6):

$$\bar{r}^2 \left( \psi - \frac{1}{\rho + \bar{\rho}|B_{j_{L_i-2},2}^n|} \right) > \psi - \frac{1}{\rho + \bar{\rho}}, \quad (5.9)$$

and

$$\bar{r}^2 \left( \psi - \frac{1}{\rho + \bar{\rho}|B_{j_{L_i-2},2}^n|} \right) > \bar{r} \left( \psi - \frac{1}{\rho + \bar{\rho}|B_{j_{L_i-2},1}^{n,\sigma^*}|} \right). \quad (5.10)$$

Furthermore, recall that we have already shown that agent  $j_{L_i-1}$  will not exit before she experiences her first communication step under any equilibrium  $\sigma^{n,*}$  provided that  $n$  is sufficiently large, which implies that  $B_{j_{L_i-1},1}^{n,\sigma^*} \subseteq B_{j_{L_i-2},2}^{n,\sigma^*}$  under any  $\sigma^{n,*}$  for sufficiently large  $n$ , and thus  $\lim_{n \rightarrow \infty} |B_{j_{L_i-2},2}^{n,\sigma^*}| \geq \lim_{n \rightarrow \infty} |B_{j_{L_i-1},1}^{n,\sigma^*}| = \infty$  under any equilibrium  $\sigma^{n,*}$ . Also we know that  $\lim_{n \rightarrow \infty} |B_{j_{L_i-2},1}^{n,\sigma^*}| < \infty$ . Together with conditions (5.9) and (5.10), these facts validate conditions (5.5) and (5.6). Hence we get that agent  $j_{L_i-2}$  will not exit before she experiences her second communication step in any  $\sigma^{n,*}$  provided that  $n$  is

sufficiently large.

The arguments above for  $j_{L_i-2}$  can be extended successively to  $j_1$ . Hence, under any equilibrium  $\sigma^{n,*}$ , no  $j_{L_i-l}$  in the established path  $\{j_{L_i-1}, j_{L_i-2}, \dots, j_1, i\}$  will exit before she experiences  $l$  communication steps under any equilibrium  $\sigma^{n,*}$  provided that  $n$  is sufficiently large. A byproduct is that  $\lim_{n \rightarrow \infty} |B_{j_{L_i-l}, l}^{n, \sigma^*}| = \infty$  under any  $\sigma^{n,*}$ , for  $l \in \{1, 2, \dots, L_i - 1\}$ .

STEP 3. Finally, we argue that the socially informed agent  $i$  will not exit before she experiences  $L_i$  communication steps under any equilibrium  $\sigma^{n,*}$  when  $n$  is sufficiently large. It requires that there exists  $N \in \mathbb{N}$  such that for all social networks  $G_n \in \{G_n\}_{n=1}^\infty$  with  $n \geq N$ , we have

$$\psi - \frac{1}{\rho + \bar{\rho}|B_{i, L_i}^{n, \sigma^*}|} > 0, \quad (5.11)$$

and

$$\bar{r}^{L_i} \left( \psi - \frac{1}{\rho + \bar{\rho}|B_{i, L_i}^{n, \sigma^*}|} \right) > \bar{r}^l \left( \psi - \frac{1}{\rho + \bar{\rho}|B_{i, l}^{n, \sigma^*}|} \right), \quad (5.12)$$

for all  $l < L_i$ .

Recall that we have already shown that agent  $j_{L_i-l}$  in the constructed path will not exit before she experiences  $L_i - l$  communication steps for  $l \in \{1, 2, \dots, L_i - 1\}$ , under any equilibrium  $\sigma^{n,*}$  provided that  $n$  is sufficiently large, which implies that  $B_{j_{L_i-1}, 1}^{n, \sigma^*} \subseteq B_{j_2, L_i-2}^{n, \sigma^*} \subseteq \dots \subseteq B_{j_1, L_i-1}^{n, \sigma^*} \subseteq B_{i, L_i}^{n, \sigma^*}$  under any  $\sigma^{n,*}$  for sufficiently large  $n$ , and thus  $\lim_{n \rightarrow \infty} |B_{i, L_i}^{n, \sigma^*}| \geq \lim_{n \rightarrow \infty} |B_{j_1, L_i-1}^{n, \sigma^*}| \geq \dots \geq \lim_{n \rightarrow \infty} |B_{j_{L_i-2}, 2}^{n, \sigma^*}| \geq \lim_{n \rightarrow \infty} |B_{j_{L_i-1}, 1}^{n, \sigma^*}| = \infty$  under any  $\sigma^{n,*}$ . Also, we have  $B_{i, l}^{n, \sigma^*} \subseteq B_{i, l}^n$  and thus  $|B_{i, l}^{n, \sigma^*}| \leq |B_{i, l}^n|$ , under any  $\sigma^{n,*}$  for  $l \in \{1, 2, \dots, L_i - 1\}$ , which implies the right hand sides of condition (4.3) are greater than or equal to the right hand sides of condition (5.12), for  $l \in \{1, 2, \dots, L_i - 1\}$ . Moreover, we know that  $\lim_{n \rightarrow \infty} |B_{i, l}^{n, \sigma^*}| \leq \lim_{n \rightarrow \infty} |B_{i, l}^n| < \infty$  for  $l \in \{1, 2, \dots, L_i - 1\}$  by the definition of  $L_i$ . Together with conditions (4.2) and (4.3) in Definition 6, these facts validate conditions (5.11) and (5.12). Hence we get that the socially informed agent  $i$  will not exit before she experiences  $L_i$  communication steps and she can enjoy a positive payoff when she experiences  $L_i$  communication steps, under any  $\sigma^{n,*}$  provided that  $n$  is sufficiently

large. This further implies  $k_i^{n,\sigma^*} \geq |B_{i,L_i}^{n,\sigma^*}|$  under any  $\sigma^{n,*}$  with sufficiently large  $n$ , which finally leads to  $\lim_{n \rightarrow \infty} |k_i^{n,\sigma^*}| \geq \lim_{n \rightarrow \infty} |B_{i,L_i}^n| = \lim_{n \rightarrow \infty} |B_{i,L_i}^n| = \infty$  under any  $\sigma^{n,*}$  when  $L_i \geq 2$ . This concludes the proof.